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a −Browder's Theorem For Conditionally Totally Posinormal Operators

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ABSTRACT

In this paper we present results on Browder's theorem for conditionally totally posinormal operators using established results on kato type operators and polaroid operators.

Keywords: Conditionally totally posinormal operators, Polaroid property, SVEP, Kato type operators

1. INTRODUCTION

Throughout this paper, $A \in B(H)$ denotes a bounded linear operator acting on an infinite dimensional Hilbert space H into itself. We consider a conditionally totally posinormal operator A such that $A - \lambda I \in Q$ i.e $\sigma(A-\lambda I)_{|M} = 0 \implies (A-\lambda I)_{|M} = 0$ and $iso\sigma(A) \cap \sigma_k(A) = \emptyset$ following lemma 3.2[1], this is a necessary and sufficient condition that ensures every isolated point of the spectrum of A is a pole of the resolvent of A. As a result the operator A is polaroid, in particular $H_\circ(A-\lambda I) = N(A-\lambda I)^n, n = n(\lambda) \in \mathbb{N}$ for all $\lambda \in iso(\sigma(A))$ (Theorem 2.9,[6]). The authors in [1], using the SVEP and Kato spectrum for operators on H present conditions under which posinormal operators satisfy Weyl's theorem. In particular it has been shown that; If $A \in (CTP)$ is such that $iso\sigma(A) \cap \sigma_k(A) = \emptyset$ then $f(A)$ satisfies Weyl's theorem for every $f \in \mathcal{H}(\sigma(A))$ (set of analytic functions which are defined on an open neighborhood U of $\sigma(A)$ and A^* satisfies a-Weyl's theorem.

2. NOTATION AND TERMINOLOGY

Throughout this paper, we shall use the following notations as used in operator theory for Hilbert space operators; let A^*; $R(A)$; $N(A)$; $\rho(A)$; $\sigma(A)$; $\sigma_p(A)$; $\sigma_{ap}(A), \sigma_k(A)$ denote respectively adjoint, Range, kernel, resolvent set, spectrum, point spectrum and approximate point spectrum.

Definition 2.1[2] $A \in B(H)$ is conditionally totally posinormal abbreviated CTP if to each $\lambda \in \mathbb{C}$ there corresponds a positive operator P_λ such that $A \in B(H)$: $|(A-\lambda I)^*|^2 = |P_\lambda^{\frac{1}{2}}(A-\lambda I)|^2$ for all $\lambda \in \mathbb{C}$.

Definition 2.2[3] $A \in B(H)$ is said to be semi-regular if $R(A)$ is closed and $N(A) \subseteq R(A^n)$ for every $n \in \mathbb{N}$ and A is Kato type at a point $\lambda \in \mathbb{C}$ if there exists a pair of A-invariant closed subspaces $(M;N)$ such that $H = M \oplus N$, the restriction $(A-\lambda I)_{|M}$ is nilpotent and $(A-\lambda I)_{|N}$ is semi-regular, i.e A admits the generalized Kato decomposition at $\lambda \in \mathbb{C}$.

The Kato spectrum of A is defined by; $\sigma_k(A) = \{\lambda \in \mathbb{C}: A - \lambda I is not kato type\}$

Definition 2.4[9] $A \in B(H)$ is said to be Fredholm if it has a finite index. The essential spectrum (or the Fredholm spectrum) is defined by;

$$\sigma_e(A) = \{\lambda \in \mathbb{C}: \lambda I - A is not Fredholm\}.$$

Let $\mathcal{W}(H)$denote the class of Fredholm operators of index zero known as Weyl operators and the Weyl spectrum denoted by$\omega(A) = \{\lambda \in C: \lambda I - A is not Weyl\}$.

Remark 2.5

Weyl's theorem holds for A if $\sigma(A)\backslash\omega(A) = \pi_{\circ\circ}(A)$ where $\pi_{\circ\circ}(A)$=$\{\lambda \in iso\sigma(A): 0 < \alpha(A - \lambda I) < 0\}$ is the set of isolated points of $\sigma(A)$ which are eigenvalues of finite multiplicity.We say that a-Weyl's theorem holds for $A \in B(H)$ if $\sigma_{ap}(A)\backslash\omega(A) = \pi_{\circ\circ}^{a}(A)$,where $\pi_{\circ\circ}^{a}(A)$denotes the set of isolated points of $\sigma_{ap}(A)$ which are eigenvalues of finite multiplicity.[11]

It has been shown in [13] that; a-Weyl's theorem ⟹Weyl's theorem

Definition 2.6 An operator A is Browder if it is Fredholm and has finite both ascent and descent. The set$\sigma_b(A) = \{\lambda \in \mathbb{C}: \lambda I - A is not Browder\}$is the Browder spectrum of A.

Evidently $\sigma_e(A) \subseteq \omega(A) \subseteq \sigma_b(A) \subseteq acc\sigma(A) \cup \sigma_e(A) \subseteq \sigma(A)$.

Remark 2.7

Browder's theorem holds if$\sigma(A)\backslash\omega(A) = p_{\circ\circ}(A)$, where$p_{\circ\circ}(A) = \sigma(A)\backslash\sigma_b(A)$. a-Browder's theorem holds for $A \in B(H)$ if$\sigma_{ap}(A)\backslash\sigma_{ab}(A) = \pi_{\circ}^{a}(A)$, where $\pi_{\circ}^{a}(A)$ is the set of all left poles of finite rank. [15]

Several authors have shown that for any $A \in B(H)$ the following implications hold;

a-Weyl's theorem⟹ Weyl's theorem ⟹ Browder's theorem [4]

a-Weyl's theorem⟹ a-Browder's theorem ⟹Browder's theorem [4]

Definition 2.8 An operator $A \in B(H)$ have Single value extension property (SVEP) if for any analytic function $f : D \to H$ with $(\lambda I - A)f(\lambda) \equiv 0$,it results $f(\lambda) \equiv 0$.

Definition 2.9[4]

The quasi-nilpotent part of $\lambda I - A$ is defined by; $H_{\circ}(\lambda I - A) \coloneqq \left\{x \in H: \lim_{n\to\infty} \|(\lambda I - A)^n\|^{\frac{1}{n}}\right\}$while the analytic core is defined as the set $K(\lambda I - A)$for all $a \in H$ such that $\exists\, c > 0$ and a sequence (a_n) in H for which$(\lambda I - A)a_1 = a, (\lambda I - A)a_{n+1} = a_n$ and $\|a_n\| \leq c^n\|a\|$for all $\in \mathbb{N}$.

3. MAIN RESULTS

Lemma 3.1. (Lemma 3.2, [3])

Let $A \in B(H)$ and let λ be an isolated point in$\sigma(A)$.Then the following properties are equivalent;

i) $\lambda$is pole of A

ii) There exists A-invariant subspaces M and N of H such that,

$(A-\lambda I)=(A-\lambda I)_{|M}\oplus(A-\lambda I)_{|N}$ on $H = M\oplus N$, where $(A-\lambda I)_{|M}$ is bounded below and $((A-\lambda I)_{|N}$ is nilpotent.

Theorem 3.2

If $A\in (CTP)$ is such that $iso\sigma(A)\cap\sigma_k(A)=\emptyset$ and $(A-\lambda I)_{|H\circ(A-\lambda I)}\in\mathcal{Q}$, then $A-\lambda I$ is Kato type.

Proof

Suppose $A\in (CTP)$ and $iso\sigma(A)\cap\sigma_k(A)=\emptyset$ then by lemma 3.1 every isolated point in $\sigma(A)$ is a pole of A and $H=H\circ(A-\lambda I)\oplus K(A-\lambda I)$ thus $A-\lambda I$ has the decomposition $(A-\lambda I)=(A-\lambda I)_{|H\circ(A-\lambda I)}\oplus(A-\lambda I)_{|K(A-\lambda I)}$ where $(A-\lambda I)_{|H\circ(A-\lambda I)}$ is bounded below and $(A-\lambda I)_{|K(A-\lambda I)}$ is nilpotent, therefore by definition 2.2, it follows that A admits the generalized Kato decomposition property and hence is of Kato type.

Lemma 3.3[5]

For a bounded operator $A\in B(H))$ Browder's theorem holds precisely when one of the following statements hold;

i) $H\circ(A-\lambda I)$ is finite-dimensional for every $\lambda\in\sigma(A)\backslash\omega(A)$

ii) $H\circ(A-\lambda I)$ is closed for all $\lambda\in\sigma(A)\backslash\omega(A)$

iii) $K(A-\lambda I)$ is finite dimensional $\lambda\in\sigma(A)\backslash\omega(A)$

Theorem 3.4.

If $A\in (CTP)$ is such that $iso\sigma(A)\cap\sigma_k(A)=\emptyset$ and $((A-\lambda I)_{|H\circ(A-\lambda I)}\in\mathcal{Q}$, then Browder's theorem holds for A.

Proof

Following from theorem 3.2, A admits the generalized Kato decomposition property. Let $\lambda\in\sigma(A)\backslash\omega(A)$ then by Kato decomposition A^* has SVEP at λ and by corollary 2.45[4], $\sigma_{ap}(A)=\sigma(A)$ and $H\circ(A-\lambda I)$ is finite dimensional by lemma 3.3. It follows from theorem 2.4.3[4] that $\lambda\in p_{\circ\circ}(A)$.

Conversely suppose that $\lambda\in p_{\circ\circ}(A)$, then since A^* has SVEP at λ then following corollary 2.10[4], $p_{\circ\circ}(A)=\pi_{\circ\circ}(A)$ and since Weyl's theorem holds for A, $\lambda\in\sigma(A)\backslash\omega(A)$.

The next result is a necessary and sufficient condition for a-Browder's theorem to hold for a conditionally totally posinormal operator established using the relationship between the SVEP and the spectral mapping theorem for Hilbert space operators.

Theorem 3.5. If $A \in (CTP)$ is such that $iso\sigma(A) \cap \sigma_k(A) = \emptyset$; and $((A-\lambda I)_{|H_\circ(A-\lambda I)} \in \mathcal{Q}$,then $f(A)$ satisfies a-Browder's theorem for every $f \in \mathcal{H}(\sigma(A))$.

Proof

a −Browder's theorem holds if $\sigma_{ab}(f(A)) = \omega_a(f(A))$,since $\sigma_{ab}(f(A)) \subseteq acc\sigma_{ap}(f(A))$,it suffices to show that $acc\sigma_{ap}(f(A)) \subseteq \omega_a(f(A))$.Since A is Kato type at every $\lambda \in \sigma_{ap}(A)$it follows that A^* has SVEP at $\lambda \in \sigma_{ap}(A)$then by the duality theorem in theorem 2.40[4], $f(A)$has SVEP.Suppose that $\lambda \notin \omega_a(f(A))$then $A - \lambda I$ is upper Fredholmand SVEP at A^*ensures that $\sigma_{ap}(f(A))$ does not cluster at λi.e.$\lambda \notin acc\sigma_{ap}(f(A))$.It has been shown in theorem 3.3 [1] that $f(A)$satisfies a-Weyl's theorem thus since A^* has SVEP, the spectral mapping theorem implies that $\sigma_{ab}(f(A)) = \omega_a(f(A))$it follows that a-Browder's theorem holds for $f \in \mathcal{H}(\sigma(A))$.

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