



A Study on Properties of (n, m) - Hyponormal Operators

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

This paper looks at the properties of (n, m) - hyponormal operators. We show that for an operator A that is (n, m) - hyponormal, and it is equivalent under an isometry to an operator B , then B is also (n, m) -hyponormal. Additionally, the concept of (n, m) -unitary quasiequivalence is introduced, and it is also shown that if an operator A is (n, m) - hyponormal, and is (n, m) -unitary quasiequivalence to an operator B , then B is also (n, m) - hyponormal.

Keywords: (n, m) -hyponormal; isometric equivalence; (n, m) -unitary quasiequivalence.

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1 Introduction

This section provides definitions and basic results that are required.

In this paper, the class of all linear operators in Hilbert space (H) that are bounded is denoted by $B(H)$. The class of hyponormal operators was first studied by Joseph Gail Stampfli [1], though Halmos [2] had looked at it under another name. Berberian [3] also looked at hyponormal operators, he particularly showed by use of approximate proper vectors that every completely continuous hyponormal must be normal. Sheith [4] looked

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at the relationship between hyponormal operators and self-adjoint operators. Aluthge extended the concept to p -hyponormal operators in 1990 [5]. Huruya [6] studied the spectral and order properties of the Aluthge transforms of p -hyponormal operators. Jibril [6] introduced the class of n -normal operators to generalize of the class of operators that are normal and studied their properties, Alzuraiqi and Patel [7] extended the results on n -normal. Abood and Al-loz [8] studied the class of (n, m) - normal operators and provided sufficient conditions for an operator to be (n, m) -power normal. In 2016, Guesba and Nadir [9], introduced the class of n -power hyponormal operators. Mecheri [10], while studying the hyperinvariant subspace problem, gave an affirmative solution on the presence of a nontrivial hyperinvariant subspace for operators that are hyponormal. He further showed that large classes of operators that contain operators that are hyponormal also have nontrivial subspaces that are hyperinvariant. It therefore follows that the class of operators that are n -hyponormal has a nontrivial subspace that is invariant. Combining results from studies done by [8] and [9], this research introduces the class of (n, m) - hyponormal operators, we present some of its properties and give its relationship with other classes of operators [11, 12].

Definition 1.1. Let H be a complex Hilbert space. A bounded operator $A \in H$ is:

1. Unitary if $A^*A = AA^* = I$
2. Isometry if $A^*A = I$
3. Co-isometry if $AA^* = I$
4. Normal if $A^*A = AA^*$
5. n -Normal if $A^*A^n = A^nA^*$
6. Hyponormal if $AA^* \leq A^*A$
7. n -Hyponormal if $A^nA^* \leq A^*A^n$

Definition 1.2. Let A be a bounded linear operator in H . Then the A is (n, m) power hyponormal if $A^n(A^m)^* \leq (A^m)^*A^n$ for some natural numbers n and m . The class containing these operators will be denoted by $[(n, m)HN]$

Remark 1.1. For an operator A that belongs to the class $[(n, m)HN]$. We have

1. If $n = m = 1$, then (n, m) -power hyponormal becomes hyponormal, i.e. $[(1, 1)HN] = [HN]$
2. If $m = 1$, then $(n, 1)$ -power hyponormal becomes n -hyponormal, i.e. $[(n, 1)HN] = [nHN]$
3. $A \in [(n, m)HN] \iff [A^n, A^{*m}] \leq 0$

Definition 1.3 (11). Two operators bounded linear operators A and B in Hilbert space (H) are unitarily equivalent if there is an operator V that is unitary such that $A = V^*BV$

Definition 1.4 (12). Two operators bounded linear operators A and B in Hilbert space (H) are unitarily quasiequivalent if there is an operator V that is unitary such that $AA^* = VBB^*V^*$ and $A^*A = VB^*BV^*$

Definition 1.5. Two operators bounded linear operators A and B in Hilbert space (H) are (n, m) -unitarily quasiequivalent if there is an operator V that is unitary such that $A^nA^{m*} = VB^nB^{m*}V^*$ and $A^{m*}A^n = VB^{m*}B^nV^*$

Theorem 1.1. Let $A \in B(H)$ be (n, m) - hyponormal operator then:

1. A is (m, n) - hyponormal operator.
2. A^k is (n, m) - hyponormal operator for $k \in N$
3. αA is (n, m) - hyponormal operator for $\alpha \in R$.
4. A^{mn} is nm -hyponormal operator.

5. A^k is hyponormal operator for k the lcm of n and m .

Proof. 1. Let A be a linear operator that is bounded and (n, m) - hyponormal, then

$$\begin{aligned} A^m(A^n)^* &= (A^n(A^m)^*)^* \\ &\leq ((A^m)^*A^n)^* \\ &= (A^n)^*A^m \end{aligned}$$

Hence A is (m, n) - hyponormal.

2. Let A be a linear operator that is bounded and (n, m) - hyponormal, then

$$\begin{aligned} (A^k)^n((A^k)^m)^* &= \underbrace{(A^n \cdots A^n)}_{k \text{ - times}} \underbrace{(A^m \cdots A^m)^*}_{k \text{ - times}} \\ &= \underbrace{(A^n \cdots A^n)}_{(k-1) \text{ - times}} A^n (A^m)^* \underbrace{(A^m \cdots A^m)^*}_{(k-1) \text{ - times}} \\ &\leq \underbrace{(A^n \cdots A^n)}_{(k-1) \text{ - times}} (A^m)^* A^n \underbrace{(A^m \cdots A^m)^*}_{(k-1) \text{ - times}} \\ &= \underbrace{(A^n \cdots A^n)}_{(k-2) \text{ - times}} A^n (A^m)^* A^n (A^m)^* \underbrace{(A^m \cdots A^m)^*}_{(k-2) \text{ - times}} \\ &= \underbrace{(A^n \cdots A^n)}_{(k-2) \text{ - times}} (A^m)^* A^n (A^m)^* A^n \underbrace{(A^m \cdots A^m)^*}_{(k-2) \text{ - times}} \\ &\vdots \\ &= ((A^m)^* A^n)^k \\ &= ((A^k)^m)^* (A^k)^n. \end{aligned}$$

Hence A^k is (n, m) -power hyponormal operator for $k \in N$.

3. Let A be a linear operator that is bounded and (n, m) - hyponormal, then

$$\begin{aligned} (\alpha A)^n(\alpha A)^m)^* &= \alpha^n A^n \overline{\alpha}^m (A^m)^* \\ &= \alpha^n \overline{\alpha}^m A^n (A^m)^* \\ &\leq \alpha^n \overline{\alpha}^m (A^m)^* A^n \\ &= \overline{\alpha}^m (A^m)^* \alpha^n A^n \\ &= (\alpha A)^m)^* (\alpha A)^n. \end{aligned}$$

Hence αA is (m, n) - hyponormal.

4. Let A be a linear operator that is bounded and (n, m) -hyponormal, then

$$\begin{aligned}
 A^{nm}(A^{nm})^* &= (A^n)^m((A^m)^n)^* \\
 &= \underbrace{(A^n \dots A^n)}_{m\text{-times}} \underbrace{(A^m \dots A^m)^*}_{n\text{-times}} \\
 &= \underbrace{(A^n \dots A^n)}_{m-1\text{ times}} A^n (A^m)^* \underbrace{(A^m \dots A^m)^*}_{n-1\text{ times}} \\
 &\leq \underbrace{(A^n \dots A^n)}_{m-1\text{ times}} (A^m)^* A^n \underbrace{(A^m \dots A^m)^*}_{n-1\text{ times}} \\
 &\vdots \\
 &= ((A^m)^*)^n (A^n)^m \\
 &= (A^{nm})^* A^{nm}.
 \end{aligned}$$

Hence A^{nm} is nm -hyponormal.

5. Assume A is a linear operator that is bounded and (n, m) -hyponormal. Let $k = pn$ and $k = qm$. We have

$$\begin{aligned}
 A^k(A^k)^* &= (A)^{pn}((A^*)^{qm}) \\
 &= [A^n]^p [(A^*)^m]^q \\
 &= \underbrace{(A^n \dots A^n)}_{p\text{-times}} \underbrace{(A^*)^m \dots (A^*)^m}_{q\text{-times}} \\
 &= \underbrace{(A^n \dots A^n)}_{p-1\text{ times}} A^n (A^m)^* \underbrace{(A^*)^m \dots (A^*)^m}_{q-1\text{ times}} \\
 &\leq \underbrace{(A^n \dots A^n)}_{m-1\text{ times}} (A^*)^m A^n \underbrace{(A^*)^m \dots (A^*)^m}_{q\text{-times}} \\
 &\vdots \\
 &= ((A^*)^{qm})(A^{np}) \\
 &= (A^{qm})^* A^{np} \\
 &= (A^k)^* A^k.
 \end{aligned}$$

Hence A^k is hyponormal.

□

2 Main Results

In this section, we discuss the equivalence of (n, m) -hyponormal operators under an isometry.

Theorem 2.1. *The relation (n, m) -unitary quasiequivalence is an equivalence relation.*

Proof. We show that the three conditions that define an equivalence relations are satisfied by the class of (n, m) -unitary quasiequivalent operators in Hilbert space.

Let A, B , and C be operators that are bounded in a Hilbert space (H) and U be a unitary operator. Clearly, A is (n, m) -unitary quasiequivalent to A since $A^{m*}A^n = IA^{m*}A^nI^*$ for $I = U$.

Let A be (n, m) -unitary quasiequivalent to B , then

$$A^{m*}A^n = UB^{m*}B^nU^*$$

and

$$A^n A^{m*} = UB^n B^{m*} U^*$$

Premultiplying and postmultiplying the preceding equation by U^* and U respectively yields

$$B^{m*} B^n = UA^{m*} A^n U^*$$

and

$$B^n B^{m*} = UA^n A^{m*} U^*$$

Hence B is (n, m) -unitary quasiequivalent to A . Lastly, suppose A is (n, m) -unitary quasiequivalent to B and B is (n, m) -unitary quasiequivalent to C , then

$$A^{m*} A^n = UB^{m*} B^n U^*$$

and

$$A^n A^{m*} = UB^n B^{m*} U^*$$

and

$$B^{m*} B^n = VA^{m*} A^n V^*$$

and

$$B^n B^{m*} = VA^n A^{m*} V^*$$

for U and V unitary operators. Then

$$A^n A^{m*} = UB^n B^{m*} U^* = UVC^n C^{m*} V^* U^* = WC^n C^{m*} W^*$$

where $W = UV$ is unitary. Also,

$$A^{m*} A^n = UB^{m*} B^n U^* = UVC^{m*} C^n V^* U^* = WC^{m*} C^n W^*$$

Thus, A is (n, m) -unitary quasiequivalent to C . Therefore, (n, m) -unitary quasiequivalence is an equivalence relation. \square

Theorem 2.2. Let $A \in B(H)$ be (n, m) -hyponormal operator and $B \in B(H)$ be such that:

1. $B = UAU^*$ with U being an isometry, then B is also (n, m) -hyponormal.
2. $B = V^*AV$ with V being a co-isometry, then B is also (n, m) -hyponormal.

Proof. 1. Let $B \in B(H)$ be such that $B = UAU^*$ for U an isometry. Then

$$\begin{aligned} B^* &= (UAU^*)^* \\ &= UA^*U^* \end{aligned}$$

and

$$\begin{aligned} B^n &= (UAU^*)^n \\ &= \underbrace{(UAU^* \cdots UAU^*)}_{n \text{ - times}} \\ &= UA^n U^* \end{aligned}$$

similarly,

$$B^m = UA^m U^*$$

Therefore,

$$\begin{aligned}
 B^n(B^*)^m &= UA^nU^*(UA^*U^*)^m \\
 &= UA^nU^*U(A^*)^mU^* \\
 &= UA^n(A^*)^mU^* \\
 &\leq U(A^*)^mA^nU^* \\
 &= U(A^*)^mU^*UA^nU^* \\
 &= (B^*)^mB^n
 \end{aligned}$$

Hence, B is (n, m) - hyponormal.

2. Let $B \in B(H)$ be such that $B = V^*AV$ for V a co-isometry. Then

$$\begin{aligned}
 B^* &= (V^*AV)^* \\
 &= V^*A^*V
 \end{aligned}$$

and

$$\begin{aligned}
 B^n &= (V^*AV)^n \\
 &= \underbrace{(V^*AV \cdots V^*AV)}_{n \text{ - times}} \\
 &= V^*A^nV
 \end{aligned}$$

similarly,

$$B^m = V^*A^mV$$

Therefore,

$$\begin{aligned}
 B^n(B^*)^m &= V^*A^nV(V^*A^*V)^m \\
 &= V^*A^nVV^*(A^*)^mV \\
 &= V^*A^n(A^*)^mV \\
 &\leq V^*(A^*)^mA^nV \\
 &= V^*(A^*)^mVV^*A^nV \\
 &= (B^*)^mB^n
 \end{aligned}$$

Hence, B is (n, m) - hyponormal. □

Corollary 2.3. *Let A be a bounded operator that is (n, m) - hyponormal and B be unitarily equivalent to A . Then B is also (n, m) - hyponormal.*

Proof. Let A be a bounded operator that is (n, m) - hyponormal, since B is unitarily equivalent to A , we have $B = U^*AU$ where U is a unitary operator. Then

$$\begin{aligned}
 B^n(B^*)^m &= U^*A^nU(U^*A^*U)^m \\
 &= U^*A^nUU^*(A^*)^mU \\
 &= U^*A^n(A^*)^mU \\
 &\leq U^*(A^*)^mA^nU \\
 &= U^*(A^*)^mUU^*A^nU \\
 &= (B^*)^mB^n
 \end{aligned}$$

Hence, B is (n, m) - hyponormal. □

Corollary 2.4. Let A, A^* be bounded operators in H that are (n, m) -hyponormal. Then A is (n, m) -normal.

Proof. Since both A and A^* are bounded operators that are both (n, m) -hyponormal, it follows that

$$A^*A \leq AA^*$$

and

$$AA^* \leq A^*A$$

From these two operator equations, it follows that

$$AA^* = A^*A$$

Hence A is (n, m) -normal. □

Theorem 2.5. Let B be (n, m) -hyponormal operator and A be (n, m) -quasiequivalent to B . Then A is (n, m) -hyponormal operator.

Proof. Since B is (n, m) -quasiequivalent to A then $B^n B^{m*} = UA^n A^{m*} U^*$ and $B^{m*} B^n = UA^{m*} A^n U^*$. Now;

$$\begin{aligned} A^n A^{m*} &= UB^n B^{m*} U^* \\ &\leq UB^{m*} B^n U^* \\ &= A^{m*} A^n. \end{aligned}$$

Hence A is (n, m) -hyponormal operator. □

Theorem 2.6. Let A and B be commuting bounded (n, m) -hyponormal operators and $AB^* = B^*A$. Then AB is (n, m) -hyponormal operator.

Proof. Since $AB = BA$ we have $A^n B^n = (AB)^n$ and $AB^* = B^*A$, implies that $A^n (B^*)^m = (B^*)^m A^n$. Now;

$$\begin{aligned} (AB)^n ((AB)^*)^m &= (A^n B^n) (A^m B^m)^* \\ &= A^n B^n (B^*)^m (A^*)^m \\ &\leq A^n (B^*)^m B^n (A^*)^m \\ &= (B^*)^m A^n T^n (A^*)^m \\ &= (B^*)^m A^n (A^*)^m B^n \\ &\leq (B^*)^m (A^*)^m A^n B^n \\ &= ((AB)^*)^m (AB)^n. \end{aligned}$$

Hence AB is (n, m) -hyponormal operator. □

Theorem 2.7. Let A_1, A_2, \dots, A_p be bounded (n, m) -hyponormal operators in a Hilbert space (H) . Then:

1. $(A_1 \oplus A_2 \oplus \dots \oplus A_p)$ and
2. $(A_1 \otimes A_2 \otimes \dots \otimes A_p)$ are (n, m) -power hyponormal

Proof. 1. Since we have

$$\begin{aligned} (A_1 \oplus \dots \oplus A_p)^n (A_1 \oplus \dots \oplus A_p)^{*m} &= (A_1^n \oplus \dots \oplus A_p^n) (A_1^{*m} \oplus \dots \oplus A_p^{*m}) \\ &= A_1^n A_1^{*m} \oplus \dots \oplus A_p^n A_p^{*m} \\ &\leq A_1^{*m} A_1^n \oplus \dots \oplus A_p^{*m} A_p^n \\ &= (A_1^{*m} \oplus \dots \oplus A_p^{*m}) (A_1^n \oplus \dots \oplus A_p^n) \\ &= (A_1 \oplus \dots \oplus A_p)^{*m} (A_1 \oplus \dots \oplus A_p)^n \end{aligned}$$

Therefore, $(A_1 \oplus \dots \oplus A_p)$ is (n, m) -power hyponormal.

2. For $x_1, x_2, \dots, x_p \in H$

$$\begin{aligned} (A_1 \otimes \dots \otimes A_p)^n (A_1 \otimes \dots \otimes A_p)^{*m} (x_1 \otimes \dots \otimes x_p) &= (A_1^n \otimes \dots \otimes A_p^n) (A_1^{*m} \otimes \dots \otimes A_p^{*m}) (x_1 \otimes \dots \otimes x_p) \\ &= A_1^n A_1^{*m} x_1 \otimes \dots \otimes A_p^n A_p^{*m} x_p \\ &\leq A_1^{*m} A_1^n x_1 \otimes \dots \otimes A_p^{*m} A_p^n x_p \\ &= (A_1^{*m} \otimes \dots \otimes A_p^{*m}) (A_1^n \otimes \dots \otimes A_p^n) (x_1 \otimes \dots \otimes x_p) \\ &= (A_1 \otimes \dots \otimes A_p)^{*m} (A_1 \otimes \dots \otimes A_p)^n (x_1 \otimes \dots \otimes x_p) \end{aligned}$$

Hence, $(A_1 \otimes \dots \otimes A_p)$ is (n, m) -power hyponormal. □

Corollary 2.8. *Let A_1, A_2, \dots, A_p be bounded n -hyponormal operators in a Hilbert space (H) . Then $(A_1 \oplus A_2 \oplus \dots \oplus A_p)$ and $(A_1 \otimes A_2 \otimes \dots \otimes A_p)$ are n -power hyponormal.*

Proposition 2.1. *If A is a $(2, 2)$ -power hyponormal operator and A is idempotent, then A is a hyponormal operator.*

Proof. Since A is $(2, 2)$ -power hyponormal operator, then $A^2 A^{*2} \leq A^{*2} A^2$ and hence $A^2 (A^2)^* = (A^2)^* A^2$. But A being idempotent implies that $A^2 = A$. Therefore, we have $AA^* \leq A^*A$. Hence, A is a hyponormal operator. □

Proposition 2.2. *If A is an operator that is both (n, m) -power hyponormal operator and idempotent, then it is hyponormal.*

Conclusion

The study introduced the class of (n, m) -hyponormal operators and the concept of (n, m) -quasiequivalence which is shown to be an equivalence relation. Several properties of (n, m) -hyponormal operators have been proved and has been particularly proved that if B is (n, m) -hyponormal operator and A be (n, m) -quasiequivalent to B , then it follows that B is also an (n, m) -hyponormal operator.

Competing Interests

Authors have declared that no competing interests exist.

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