BACHELOR OF DATA SCIENCE

MAT 111

Foundations of Mathematics

LEARNING

MODULE 9

Counting: Permutations and Combinations

The Open University of Kenya

$\phi := \frac{1+\sqrt{-3}}{2}$

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Module Description

This module introduces develops techniques of determining, without direct enumeration, the number of elements in a set, or the number of possible outcomes of an event. Such counting includes the study of permutations and combinations.

In this module, you will be introduced to the following concepts:

- Basic counting principles,
- The binomial theorem and Pascal's triangle,
- Permutations and combinations,
- The Pigeonhole principle and its applications.

Expectations

This module seeks to enable learning about:

1. state the two basic counting principles,
2. determine the number of permutations of objects,
3. Apply the binomial theorem to calculating combinations
4. Apply the Pigeonhole principle to solving problems.

Module Learning Outcomes

By the end of this module, you should be able to:

1. state the two basic counting principles,
2. relate sampling of objects to concepts in permutations,
3. apply concepts of combinations to objects,
4. explain the pigeonhole principle and its applications.

Warm up

Hello and welcome, you are now at the first part of this module. Your first task is to measure your prior knowledge and the concepts to be mastered by solving the given problems.

1. Computational questions

All these questions will be randomised STACK questions. What we give below is just a small sample

(a) If I only want to arrange three of the five books on my shelf, how many ways are there to do that?

(b) Suppose a bookcase shelf has 5 English texts, 3 Chemistry texts, 6 Maths texts, and 4 Physics texts. Find the number n of ways a student can choose:

  i. one textbook from the shelf, A Chemistry textbook,

  ii. A maths textbook and a Physics text book,

  iii. One textbook for each subject

(c) If 10 people participate in a race and only the first three places are recorded, how many possible results are there?

(d) How many functions $f : \{1, 2, 3\} \to \{1, 2, 3\}$ are bijections?

(e) Two of your ten friends, Davis and Anne just broke up. They can't stand to be in a room together. How many ways are there to choose five out of ten friends to invite to dinner, ensuring that Davis and Anne are not both invited?

Self Pre-recorded Video 1

There is a lecture to be recorded covering concepts on: the basic counting principles which are the sum rule principle and the product rule principle, the factorial function and its application in the binomial theorem, the binomial theorem and Pascal's triangle.

Counting principles

There are two counting principles that will be used in this module:

Sum Rule Principle: Suppose some event E can occur in m ways and a second event F can occur in n ways, and suppose both events cannot occur simultaneously. Then E or F can occur in $m + n$ ways.

The set theoretic interpretation of the sum rule is: suppose $n(A)$ denotes the number of elements in set A and suppose that sets A and B are disjoint. Then

$$n(A \cup B) = n(A) + n(B).$$

Product Rule Principle: Suppose there is an event E which can occur in m ways and, independent of this event, there is a second event F which can occur in n ways. Then combinations of E and F can occur in $m \times n$ ways.

The set theoretic interpretation of the product rule is: suppose $n(A)$ denotes the number of elements in set A and let $A \times B$ denote the cartesian product of A and B. Then

$$n(A \times B) = n(A)n(B).$$

Example 1: Counting principles

Suppose a college has 4 different math courses, 6 different Computer courses, and 2 different psychology courses.

✍ Solution

1. The number n of ways a student can choose just one of the courses is:
$$n = 4 + 6 + 2 = 12$$
2. The number m of ways a student can choose one of each kind of courses is:
$$m = 4 \cdot 6 \cdot 2 = 48$$

The factorial notation and the binomial coefficients

The factorial of a natural number n denoted $n!$ is a function that assigns to a natural number n the product of all natural numbers less than or equal to n. That is

$$n! = n(n-1)(n-2)(n-3) \cdot 3 \cdot 2 \cdot 1$$

Example 2: The factorial function

The factorial of some selected numbers is

1. $3! = 3 \cdot 2 \cdot 1 = 6$
2. $0! = 1$
3. $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$

The **Binomial coefficient** denoted as ${}^{n}C_{k} = \binom{n}{k}$, is defined by the expression:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

This formula gives the number of ways k elements can be chosen from a set of n elements where order does not matter.

Example 3: The binomial coefficient

Evaluate $\binom{11}{7}$

$$\binom{11}{7} = \frac{11!}{7!(11-7)}! = \frac{11 \cdot 10 \cdot 9 \cdot 8 \cdot 7!}{7! \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 330$$

The **Binomial Theorem** The coefficients of the successive powers of a + b can be arranged in a triangular array of numbers, called Pascal's triangle. The powers of a + b are given by the formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

In the Pascals triangle, the first and last numbers in each row are 1. Every other number can be obtained by adding the two numbers appearing above it.

Example 4: The Binomial Theorem and Pascals Triangle

Expand using the binomial theorem and Pascal's triangle: $(2x-3)^4$

Solution

$$\begin{aligned} (2x-3)^4 &= \sum_{k=0}^{4} \binom{4}{k} (2x)^{4-k}(-3)^k \\ &= \binom{4}{0}(2x)^4(-3)^0 + \binom{4}{1}(2x)^3(-3)^1 + \binom{4}{2}(2x)^2(-3)^2 + \binom{4}{3}(2x)^1(-3)^3 + (\\ &= 16x^4 - 96x^3 + 216x^2 - 216x + 81 \end{aligned}$$

Permutations

A **permutation** is an arrangement in a definite order of a number of objects taken some or all at a time.

Example 5: Four digit pin

In how many ways can a four digit pin be created from numbers 0, 1, - - - , 9?

Solution

Here there is no particular order in which the numbers appear. So the number of ways is

$$^{10}P_4 = \frac{10!}{(10-4)!} = 5040.$$

If the numbers in the pin can be repeated, how many ways are there of creating the pin?

We have used the formula

$$^{n}P_{k} = \frac{n!}{(n-k)!},$$

for evaluating the permutation of k objects taken from n objects.

Example 6: Permutations with repetitions

Find the number m of nine-letter words that can be formed using the letters of the word **REPRESENT**

Solution

Notice that 'R' has been repeated twice and 'E' is repeated three times! So we calculate the permutations of nine objects with these repetitions. That is

$$P(9;3,2) = \frac{9!}{3!\cdot 2!} = 30240.$$

Combinations

A **combination** is all about grouping. The number of different groups which can be formed from the available things can be calculated using combinations.

Example 7: Team selection

The Kenyan football team has 19 players preparing to play a game against Egypt. The coach will have to select 11 players to start the game. In how many ways can the starting lineup team be selected?

Solution

Here there can be no repetition of selected players (it does not make sense to select a player twice). The combination of $'k'$ persons from the available $'n'$ persons is given by the formula

$$^{n}C_{k} = \frac{n!}{k!(n-k)!}.$$

Using the above formula,

$$^{19}C_{11} = \frac{19!}{11!(19-11)!} = 75582.$$

Example 8: Animal selection

A farmer buys 2 cows, 4 sheep, and 11 hens from a man who has 8 cows, 9 sheep, and 13 hens. Find the number m of choices that the farmer has.

Solution

The farmer can choose the cows in ${}^{8}C_2$ ways, the sheep in ${}^{9}C_4$ ways and the hens in ${}^{13}C_{11}$ ways.

$$m = \binom{8}{2}\binom{9}{4}\binom{13}{11} = 28 \cdot 126 \cdot 78 = 275184.$$

How can we understand the difference between permutations and combinations.

- Permutations are used when order/sequence of arrangement is needed. Combinations are used when only the number of possible groups are to be found, and the order/sequence of arrangements is not needed.
- Permutations are used for things of a different kind. Combinations are used for things of a similar kind.
- The permutation of two things from three given things 1, 2, 3 is 12, 21, 23, 32, 13, 31. The combination of two things from three given things 1, 2, 3 is 12, 13, 23.
- For different possible arrangement of things ${}^{n}P_k = \frac{n!}{(n-k)!}$. For different possible selection of things ${}^{n}C_k = \frac{n!}{k!(n-k)!}$.
- For a given set of n and r values, the permutation answer is larger than the combination answer. This can be seen from the formulas for permutation and combination!

Quiz *This will be a randomized STACK exercise/quiz examining covered concepts.*

1. Suppose a bookcase shelf has 5 Maths texts, 3 Chemistry texts, 6 Biology texts, and 4 History texts. Find the number n of ways a student can choose: (a) one of the texts; (b) one of each type of text.
2. A committee of 5 members is to be formed with 2 male members and 3 female member. Find the number of ways in which this committee can be formed from 10 male members and 5 female members.
3. There are 10 marbles in a bag, numbered from 0 to 9. How many ways of 5 different digits could be formed by picking them up from the bag, without replacement?
4. Find the number n of distinct permutations that can be formed from all the letters of each word: (a) THESE; (b) UNUSUAL; (c) REPRESENTATION.
5. A class contains 20 students with 12 men and 8 women. Find the number n of ways to:
   (a) Select a 6-member committee from the students.
   (b) Select a 6-member committee with 3 men and 3 women.
   (c) Elect a president, vice president, and treasurer.
6.

Self Pre-recorded Video 2

There is a lecture to be recorded covering concepts on the Pigeonhole principle. Here examples are to be given illustrating the applications of these principles in real life situations.

The Pigeonhole principle

This principle seems almost obvious, but is very important in proving many statements in combinatorics. To illustrate the principle: Suppose that a flock of 10 pigeons flies into a set of 9 pigeonholes to roost. Because there are 10 pigeons but only 9 pigeonholes, at least one of these 9 pigeonholes must have at least two pigeons in it. To see why this is true, note that if each pigeonhole had at most one pigeon in it, at most 9 pigeons, one per hole, could be accommodated. The **pigeonhole principle** states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons.

The Pigeonhole principle: If $n + 1$ objects (pigeons) are put into n boxes (pigeonholes), then at least one box contains two or more objects.

Another form of the Pigeonhole principle is;

The Pigeonhole principle: If n boxes (pigeonholes) are occupied by $kn + 1$ or more objects (pigeons), where k is a positive integer, then at least one box (pigeonhole) is occupied by $k + 1$ or more objects (pigeons).

Example 9:

A bag contains 10 red balls, 10 white balls, and 10 blue balls. What is the minimum number of balls you have to choose randomly from the bag to ensure that we get 4 balls of same color?

Solution

Apply pigeonhole principle. The number of colors (pigeonholes) $n = 3$. The number of balls (pigeons) $K + 1 = 4$. Therefore the minimum number of balls required is $= Kn + 1$. By simplifying we get $Kn + 1 = 10$. Verification: If we take 3 balls of each color, we have $3 \times 3 = 9$. We now need only one ball so that 4 balls will be of the same color.

Example 10:

In a computer science department, a student club can be formed with either 10 members from first year or 8 members from second year or 6 from third year or 4 from fourth year. What is the minimum number of students we have to choose randomly from the department to ensure that a student club is formed?

✍ Solution

Here the minimum number is $(10+8+6+4)-4+1 = 25$. Verification: Suppose we have 9 first year 7 second year 5 third year and 3 fourth year students, then the total number will be 24. But a club cannot be formed from these students! By adding one more student (from any year) we can form a club.

Quiz ***This will be a randomized STACK exercise/quiz examining on the Pigeonhole principle.***

1. Find the minimum number n of elements that one needs to take from the set S $= \{1, 2, 3, ..., 9\}$ so that;
   (a) Two of the numbers add up to 10.
   (b) The sum of two of the n integers is even.
   (c) The difference of two of the n integers is 5.
2. Find the minimum number of students in a class to be sure that three of them are born in the same month.
3. Find the minimum number of students needed to guarantee that five of them belong to the same class (Form 1, Form 2, Form 3, Form 4).

RECOMMENDED SUPPLEMENTARY READING

1. Read pages [1, P 107-110] on counting principles and permutations and combinations.
2. Read pages [3, P 68-73] on Counting permutations and combinations of objects.
3. Read pages [5, P 288-290] on the fundamental counting principle and permutations and combinations.

Content Curated Video

- Permutations and Combinations https://youtu.be/XJnIdRXUi7A.
- The counting principle https://youtu.be/549eLWIu0Xk.
- The Binomial theorem and Pascals triangle https://youtu.be/s19dWIHficY.
- The Pigeonhole principle https://youtu.be/2-mxYrCNX60.
- Permutations and combinations https://youtu.be/1CTzx89Kzy4.

ACTIVITIES

Solve the following situational problems:

Activity 1: Practical Life Problem

Consider a tournament with 10 teams where each team plays against every other team. Suppose each team wins at least once. Show that at least 2 of the teams have the same number of wins.

Activity 2: Practical Life Problem

A female student is to answer 10 out of 13 questions. Find the number of her choices where she must answer:

1. the first two questions;
2. the first or second question but not both;
3. exactly 3 out of the first 5 questions;
4. at least 3 of the first 5 questions.

WRAP UP

Summary

To wrap-up, answer the following questions:

1. What is the difference between a permutation and a combination?.
2. Why Is the relationship between the Binomial theorem and Pascals Triangle?
3. What does the Pigeonhole principle imply when we have more objects that the number of boxes in which the objects are to be placed?
4. Between a permutation of k objects from a list of n objects and a combination of k objects from a list of n objects, which is larger? Give the formula for calculating permutations and a formula for calculating combinations and use them to illustrate your answer above.

TAKE HOME

Hello again, you are now at the last part of this module. Your task is to do a self-reflection on your learning experience.

Permutations and combinations appear in every aspect of our lives. From choosing objects from a given list to arranging objects in a given order. Think of a situation in which you have to choose objects, or in which you have to arrange objects in a given manner. How would you apply concepts learned in this module to that situation?

DISCUSSION AND REFLECTION

Suppose you want to invite friends to an evening Nyama Choma party. Alice and Jane are sworn enemies but can be at the party together but not sit next to each other on a table. Allan had a falling out with Christine and cannot be in one event at the same time. Titus has a legal case against Juma and cannot be in the same sitting. All these are your friends, and your 8 other friends have no issue with the six individuals. If you are organizing a party of 10, how many ways can you invite your friends? Discuss all scenarios!

Synchronous

WEEKLY ONLINE LIVE FACILITATED TUTORIAL

SESSION 4: Permutations and Combinations

5 minute review

Recap the definition of a permutation, a combination and their formulas. The Pigeonhole principle and its applications.

All questions here will be STACK questions with values randomized to aid in mastery of the concepts and to promote multiple attempts

Problems. Choose from the questions below

1. Counting principles

A Mathematics class contains 18 male students and 19 female students. Find the number n of ways that the class can elect:

1. one class representative
2. two class representatives, one male and one female,
3. one president and one vice president

2. Binomial theorem

Use the binomial theorem to

1. determine the expansion of $(x - 3)^3$ and hence calculate9997^3
2. determine the expansion of $(x + 1)^4$ and hence calculate10001^4
3. find the member of the binomial expansion of $(2x^3 + x^{-1})^{10}$ that contains x^6?
4. find the fourth member of $(x - \frac{3}{x})^9$ after expansion.

3. Permutations

Consider the three-digit numbers that can be formed from the digits 1, 2, 3, 4, and 5 with no repetition of digits allowed.

1. How many of these are even numbers?

2. How many are odd numbers?
3. How many are greater than 250?

4. Permutations.

The president of the Math and Computer Club would like to arrange a meeting with six attendees, the president included. There will be three computer science majors and three math majors at the meeting. How many ways can the six people be seated at a circular table if the president does not want people with the same majors to sit next to one other?

5. Permutations and Combinations

Use the formulas for permutation and combination to solve the following problems:

1. A class of twelve computer science students are to be divided into three groups of 3, 4, and 5 students to work on a project. How many ways can this be done if every student is to be in exactly one group?
2. The congressional committees on mathematics and computer science are made up of five representatives each, and a congressional rule is that the two committees must be disjoint. If there are 385 members of congress (representatives), how many ways could the committees be selected?
3. Suppose 10 people apply for three, identical jobs.
   (a) In how many orders can three of these 10 people be hired?
   (b) How many ways are there to permute these three people?
   (c) If the first problem counted the number of ways to order three (3) of ten (10) people and the second problem counted the number of ways any particular three can be ordered, how can these be combined to count the number of ways to select three (3) out of ten (10) people?

6. The Pigeonhole principle

A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white balls. What is the minimum number of balls we have to choose randomly from the box to ensure that we get 9 balls of same color?

7. The pigeonhole principle

Jane and Joan are very messy sisters. Their dresser drawer consists of 43 white socks, 2 black socks, 23 blue socks and 8 red socks. What is the minimum number of socks they must remove from the drawer, in order to be certain that they have removed four socks of the same color?

CORE READING AND REFERENCES

References

[1] S. Lipschutz and M. Lipson, 2021, Schaum's Outline of Discrete Mathematics, Fourth Edition (Schaum's Outlines) 4th Edition, McGraw Hill, ISBN-13: 978-1264258802

[2] Rich, B. and Thomas, C., 2017, Schaum's Outline of Geometry, Sixth Edition, McGraw-Hill Education. ISBN-9781260010572.

[3] R. T. White and A.T. Ray, 2021, Practical Discrete Mathematics: Discover math principles that fuel algorithms for computer

[4] Kenneth Rosen, 2018, Discrete Mathematics and Its Applications 8th Edition, Mc Graw Hill, ISBN- 13: 978-1260091991

[5] M. R. Spiegel and R. E. Moyer, 2019, Schaum's Outline of College Algebra 5th Ed, McGraw-Hill Education, ISBN 978-1-260-12076-9

[6] Serge Lang, 2000, Basic Mathematics, Adison-Wesley. ISBN 978-1-4612-1027-6