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Characteristic functions of some contraction operators

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Abstract

In this paper, we characterize some contraction operators in terms of their Sz.-Nagy-Foias model characteristic functions and their associated Toeplitz operators.
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Keywords: Contractive, pure isometry, pure co-isometry, defect operators, characteristic function, unitary, completely non-unitary.

Introduction

Let $\mathcal{H}$ denote a Hilbert space and $B(\mathcal{H})$ denote the Banach algebra of bounded linear operators. If $T \in B(\mathcal{H})$, then T^* denotes the adjoint of T, while $Ker(T), Ran(T), \overline{\mathcal{M}}$ and $\mathcal{M}^\perp$ stands for the kernel of T, range of T, closure of $\mathcal{M}$ and orthogonal complement of a closed subspace $\mathcal{M}$ of $\mathcal{H}$, respectively. We denote by $\sigma(T), \| T \|$ and $W(T)$, the spectrum, norm and numerical range of T, respectively.

By $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\partial\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, we denote the open unit disk and the unit circle on the complex plane, respectively.

An operator $T \in B(\mathcal{H})$ is
unitary if $T^*T = TT^* = I$.
an *isometry* if $T^*T = I$.
a *co-isometry* if $TT^* = I$.
a *contraction* if $\| T \| \leq 1$.
nilpotent if $T^n = 0$ for some positive integer n. The least such n is called the *order* of nilpotence of T.

Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be *similar* (denoted $A \sim B$) if there exists an invertible operator $N \in B(\mathcal{H}, \mathcal{K})$ such that $NA = BN$ or equivalently $A = N^{-1}BN$, and are unitarily equivalent (denoted by $A \cong B$)if there exists a unitary operator $U \in B_+(\mathcal{H}, \mathcal{K})$ (Banach algebra of all invertible operators in $B(\mathcal{H})$) such that $UA = BU$ (i.e. $A = U^*BU$, equivalently, $A = U^{-1}BU$). Two operators $A \in B(\mathcal{H})$ and $B \in B(\mathcal{K})$ are said to be metrically equivalent (denoted by $A \sim_m B$) if $\| Ax \| = \| Bx \|$, (equivalently, $|\langle Ax, Ax\rangle|^{\frac{1}{2}} = |\langle Bx, Bx\rangle|^{\frac{1}{2}}$ for all $x \in \mathcal{H}$). The concept of metric equivalence of operators was introduced in [7]. Clearly similarity, unitary equivalence and metric equivalence are equivalence relations on $B(\mathcal{H})$.

Given a contraction $T \in B(\mathcal{H})$, both $(I - T^*T)$ and $(I - TT^*)$ are positive operators and hence have unique square roots. We define $D_T = (I - T^*T)^{\frac{1}{2}}$ and $D_{T^*} = (I - TT^*)^{\frac{1}{2}}$ and call them the *defect operators* of T. Clearly, $0 \leq D_T, D_{T^*} \leq I$. The closures of their ranges $\mathcal{D}_T = \overline{D_T(\mathcal{H})}$ and $\mathcal{D}_{T^*} = \overline{D_{T^*}(\mathcal{H})}$ are called the *defect spaces* of T. The respective dimensions(ranks) d_T and d_{T^*} are called the *defect indices* of T. A contraction $T \in B(\mathcal{H})$ is said to be *completely non-unitary*(c.n.u) if there exists no nontrivial reducing subspace of $\mathcal{M} \subset \mathcal{H}$ of T on which T acts unitarily (i.e. $T|_{\mathcal{M}}$ is unitary), or equivalently if its unitary part acts on the zero space $\{0\}$. Clearly, a contraction $T \in B(\mathcal{H})$ is c.n.u if T restricted to every (non-zero T-reducing) subspace of $\mathcal{H}$ is non-unitary.

It was observed in [5] that $d_T = 0$ characterizes isometric operators, whereas $d_T = d_{T^*} = 0$ characterizes unitary operators and that these numbers measure, in a sense, the deviation of the contraction T from being unitary.
For any contraction $T \in B(\mathcal{H})$ there exists a unique decomposition $\mathcal{H} = \mathcal{M}_0 \oplus \mathcal{M}_1$ such that $T\mathcal{M}_0 \subseteq \mathcal{M}_0$ and $T\mathcal{M}_1 \subseteq \mathcal{M}_1$ and $T_0 = T|_{\mathcal{M}_0}$ is unitary, whereas $T_1 = T|_{\mathcal{M}_1}$ is completely non-unitary (c.n.u).
Thus T has a unique decomposition as $T = T_0 \oplus T_1$. This is the Nagy-Foia$s$-Langer Decomposition (see) [4]. With this decomposition, we observe that $D_T = 0 \oplus D_{T_1} = D_{T_1}$ and $D_{T^*} = 0 \oplus D_{T_1^*} = D_{T_1^*}$ (where the second equality denotes "identified with") and $\mathcal{D}_T = \mathcal{D}_{T_1}$ and $\mathcal{D}_{T^*} = \mathcal{D}_{T_1^*}$. Obviously T is unitary if and only if $\mathcal{D}_T = \mathcal{D}_{T_0} = \{0\}$.

Considering the asymptotic behaviour of contractions, Sz.-Nagy and Foias [5] introduced the following classes:
A contraction $T \in B(\mathcal{H})$ is of class:
$C_{1.}$ if $\| T^n x \| \nrightarrow 0$, strongly as $n \to \infty$ for every $0 \neq x \in \mathcal{H}$.
$C_{.1}$ if $\| T^{*n} x \| \nrightarrow 0$, strongly as $n \to \infty$ for every $0 \neq x \in \mathcal{H}$.
$C_{0.}$ if $\| T^n x \| \to 0$, strongly as $n \to \infty$ for every $x \in \mathcal{H}$.
$C_{.0}$ if $\| T^{*n} x \| \to 0$, strongly as $n \to \infty$ for every $x \in \mathcal{H}$.
C_{ij} if $T \in C_{i.} \cap C_{.j}, i, j = 0,1$.

An operator $T \in B(\mathcal{H})$ is called a proper (or pure, strict) contraction if $\| Tx \| < \| x \|$, for every $x \in \mathcal{H}$. An operator-valued analytic function $\Theta: \mathbb{D} \longrightarrow B(\mathcal{H}, \mathcal{K})$ is said to be contractive if $\| \Theta(\lambda)x \| \leq \| x \|$, for all $x \in \mathcal{H}$ and $\lambda \in \mathbb{D}$ and purely contractive if $\| \Theta(0)y \| < \| y \|$, for all $0 \neq y \in \mathcal{H}$.

2. Main Results

To every completely non-unitary (c.n.u) contraction $T \in B(\mathcal{H})$, Sz.-Nagy and Foia$s$ [5] associated a contraction-valued holomorphic function Θ_T on the open unit disk $\mathbb{D} = \{\lambda \in \mathbb{C}: |\lambda| < 1\}$ on the complex plane. Such a function $\Theta_T(\lambda): \mathcal{D}_T \longrightarrow \mathcal{D}_{T^*}$ defined for all $\lambda \in \mathbb{D}$ is given by

$$\Theta_T(\lambda) = \{-T + \lambda D_{T^*}(I_{\mathcal{H}} - \lambda T^*)^{-1} D_T\}|_{\mathcal{D}_T}$$

This means that the values of $\Theta_T(\lambda)$ can be regarded as bounded operators from $\mathcal{D}_T$ into $\mathcal{D}_{T^*}$. We note that the characteristic function of a c.n.u. contraction T is equivalent to the function in $\mathbb{H}^\infty(B(\mathcal{D}_T, \mathcal{D}_{T^*})$ given by the same formula.

For any c.n.u. contraction $T \in B(\mathcal{H})$, it is clear that $TD_T = D_{T^*}T$.

Remark. The Neumann series for $(I - \lambda T)^{-1}$ is $(I - \lambda T)^{-1} = I + \sum_{n=1}^{m-1} \lambda^n T^n$ for every $\lambda \in \mathbb{C}$. Since $\lambda \in \mathbb{D}$, we have $|\lambda| < 1$. Clearly since T is a contraction $\| \lambda T \| = |\lambda| < 1$, the Neumann series converges.

The characteristic function of a c.n.u contraction T is defined for the values λ for which the operator $(I - \lambda T^*)$ is boundedly invertible.
We say that two operators $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ and $B \in B(\mathcal{K}_1, \mathcal{K}_2)$ coincide if there exist unitary operators $U: \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ and $V: \mathcal{K}_1 \longrightarrow \mathcal{K}_2$ such that $VAU = B$. Operator coincidence is an equivalence relation which is weaker than unitary equivalence. The operator-valued functions $\Theta(\lambda): \mathbb{D} \longrightarrow B(\mathcal{H}_1, \mathcal{H}_2)$ and $\Psi(\lambda): \mathbb{D} \longrightarrow B(\mathcal{K}_1, \mathcal{K}_2)$ are said to coincide (denoted by $\Theta \cong \Psi$) if there exist unitary operators $U: \mathcal{H}_1 \longrightarrow \mathcal{K}_1$ and $V: \mathcal{H}_2 \longrightarrow \mathcal{K}_2$ such that $V\Theta(\lambda) = \Psi(\lambda)U$, for all $\lambda \in \mathbb{D}$. Moreover, for a given $B(\mathcal{H}_1, \mathcal{H}_2)$-valued purely contractive analytic function $\Theta(\lambda)$ defined on $\mathbb{D}$, there exists a c.n.u. contraction T on some Hilbert space such that $\Theta_T(\lambda)$ coincides with $\Theta(\lambda)$.

The mapping

$$\Theta_T: \mathbb{D} \longrightarrow B(\mathcal{D}_T, \mathcal{D}_{T^*})$$

is a contraction-valued (i.e. $\sup_{\lambda \in \mathbb{D}} \| \Theta_T(\lambda) \| \leq 1$), analytic function and by Wu [8], $\Theta_T(0) = -T|_{\mathcal{D}_T}$ is a pure contraction.
With the decomposition of a contraction $T \in B(\mathcal{H})$ given above Θ_T decomposes as $\Theta_T(\lambda) = \Theta_u(\lambda) \oplus \Theta_0(\lambda)$, where Θ_u is a unitary constant (i.e. if $\Theta(0)$ is a unitary operator from $\mathcal{H}$ onto $\mathcal{H}_*$, then $\Theta(\lambda) = \Theta(0)$ for all $\lambda \in \mathbb{D}$, function Θ is called a unitary constant) and Θ_0 is pure (and hence called the pure part of Θ), i.e. $\| \Theta_0(0)x \| < \| x \|$ for all $0 \neq x \in \mathcal{H}$. Thus, for all $\lambda \in \mathbb{D}$, $\Theta(\lambda)\mathcal{M}_0 \subseteq \mathcal{M}_0$ and $\Theta(\lambda)\mathcal{M}_1 \subseteq \mathcal{M}_1$.
For every contractive function $\Theta(\lambda): \mathbb{D} \longrightarrow B(\mathcal{D}_T, \mathcal{D}_{T^*})$, there exists a unique decomposition $\mathcal{D}_T = \mathcal{D}_{T_0} \oplus \mathcal{D}_{T_1}$, $\mathcal{D}_{T^*} = \mathcal{D}_{T_0^*} \oplus \mathcal{D}_{T_1^*}$, so that for every fixed λ, $\Theta_0(\lambda) := \Theta(\lambda)|_{\mathcal{D}_{T_0}}$ has range in $\mathcal{D}_{T_0^*}$ and that $\Theta_1(\lambda) := \Theta(\lambda)|_{\mathcal{D}_{T_1}}$ has range in $\mathcal{D}_{T_1^*}$ and that $\Theta_0(\lambda): \mathbb{D} \longrightarrow B(\mathcal{D}_{T_0}, \mathcal{D}_{T_0^*})$ is a unitary constant and that $\Theta_1(\lambda): \mathbb{D} \longrightarrow B(\mathcal{D}_{T_1}, \mathcal{D}_{T_1^*})$ is a purely contractive function.

Theorem 2.1 [5] *Let* $T \in B(\mathcal{H})$ *be a c.n.u contraction. Then* $\| \Theta_T(0)x \| < \| x \|$ *for all* $0 \neq x \in \mathcal{D}_T$.

Theorem 2.2 [5] *Two contractions* $T_1 \in B(\mathcal{H}_1)$ *and* $T_2 \in B(\mathcal{H}_2)$ *are unitarily equivalent if their characteristic functions* Θ_{T_1} *and* Θ_{T_2} *coincide.*

Proof. Suppose $T_2 = UT_1U^*$, for some unitary operator $U \in B(\mathcal{H}_1, \mathcal{H}_2)$. By definition, $D_{T_2} = VD_{T_1}$ and $D_{T_2^*} = WD_{T_1^*}$ and $\Theta_{T_2}(\lambda) = W\Theta_{T_1}(\lambda)V^*$, where $V = U|_{D_{T_1}}$, and $W = U|_{D_{T_1^*}}$. Clearly V and W are unitary. This establishes the claim.
The converse of Theorem 2.2 is not true in general.
The characteristic function of a c.n.u contraction T is said to be constant if $\Theta_T(\lambda) = \Theta_T(0)$, for every $\lambda \in \mathbb{D}$.

Theorem 2.3 *If* $T \in B(\mathcal{H})$ *is an isometry then the characteristic function* Θ_T *is identically zero almost everywhere.*

Proof. We first prove the result for a unitary T. For a unitary operator T, $D_T = D_{T^*} = 0$ and $\mathcal{D}_T = \overline{D_T\mathcal{H}} = \{0\}$. Therefore, $\Theta_T = -T|_{\mathcal{D}_T} = -T|_{\{0\}} = 0$, for all $\lambda \in \mathbb{D}$. Since $D_T = 0$ for a unilateral shift T, we have $\mathcal{D}_T = \{0\}$ we also have $\Theta_T = 0$. That is, $\Theta_T = 0 \in B(\mathcal{D}_T, \mathcal{D}_{T^*}) = B(\{0\}, Ker(T))$. Since by the von Neumann-Wold decomposition an isometry is a direct sum of a unitary and a unilateral shift, the result follows.

Remark. Note that in the proof of Theorem 2.3 Θ_T is constant since $\Theta_T(0) = \Theta_T(\lambda)$. From Theorem 2.3 we conclude that if T is an isometry, then $\Theta_T(0) = \Theta_T(\lambda)$, for $\lambda \in \mathbb{D}$. That is, Θ_T is constant.

Theorem 2. 4 *Let* $T \in B(\mathcal{H})$. *Then* $\Theta_{kT}(\lambda) = k\Theta_T(\bar{k}\lambda), \lambda \in \mathbb{D}$ *holds for any* $k \in \partial\mathbb{D}$.

Proof. By a simple computation and using the definition we have

$$\Theta_{kT}(\lambda) = [-kT + \lambda(I - k\bar{k}TT^*)^{\frac{1}{2}}(I - (\bar{k}\lambda)T^*)^{-1}(I - \bar{k}kT^*T)^{\frac{1}{2}}]|_{\mathcal{D}_T}.$$

Since $k \in \partial\mathbb{D}, k = e^{in\theta}, 0 \leq \theta \leq 2\pi$ and $n = 0, \pm1, \pm2, \ldots$ Thus $\bar{k}k = 1$. Thus the previous equation becomes

$$\Theta_{kT}(\lambda) = [-kT + \lambda(I - TT^*)^{\frac{1}{2}}(I - (\bar{k}\lambda)T^*)^{-1}(I - T^*T)^{\frac{1}{2}}]|_{\mathcal{D}_T}.$$

Similarly,

$$k\Theta_T(\bar{k}\lambda) = [-kT + (k\bar{k})\lambda(I - TT^*)^{\frac{1}{2}}(I - (\bar{k}\lambda)T^*)^{-1}(I - T^*T)^{\frac{1}{2}}]|_{\mathcal{D}_T}.$$

Once again, since $k \in \partial\mathbb{D}, \bar{k}k = 1$. Thus, the above equation simplifies to

$$k\Theta_T(\bar{k}\lambda) = [-kT + \lambda(I - TT^*)^{\frac{1}{2}}(I - (\bar{k}\lambda)T^*)^{-1}(I - T^*T)^{\frac{1}{2}}]|_{\mathcal{D}_T}.$$

This completes the proof.

Theorem 2.5 [5] Two c.n.u contractions T_1 and T_2 are unitarily equivalent if and only if their characteristic functions coincide.

Proof. Suppose T_1 and T_2 are unitarily equivalent c.n.u contractions. Then there exists a unitary operator U such that $T_1 = U^*T_2U$. Using the definition of the characteristic function and the fact that we have

$$\begin{aligned}
\Theta_{T_1}(\lambda) &= \Theta_{U^*T_2U}(\lambda) \\
&= -(U^*T_2U) + \lambda[I - (U^*T_2U)(U^*T_2^*U)](I - \lambda U^*T_2^*U)^{-1}(I - (U^*T_2^*U)(U^*T_2U)) \\
&= -(U^*T_2U) + \lambda[I - U^*(T_2T_2^*)U](I - \lambda U^*T_2^*U)^{-1}(I - U^*(T_2^*T_2)U) \\
&= U^*\Theta_{T_2}(\lambda)U, \ \lambda \in \mathbb{D}.
\end{aligned}$$

Without loss of generality, we let $V = U^*$. The proof of the converse is trivial.

The above result can also be proved using the Neumann series and some computation as follows:

$$\Theta_{UTU^*}(\lambda) = U[-T + \sum_{k=0}^{n-1} \lambda^{k+1} D_{T^*}{T^*}^k D_T]U^*|_{\mathcal{D}_T} = U\Theta_T(\lambda)U^*,$$

for some unitary operator U and a c.n.u. contraction T.
This result also says that if T and S are unitarily equivalent c.n.u contractions, then their characteristic functions Θ_T and Θ_S are unitarily equivalent, with the unitary equivalence being implemented by the same operator U.

Remark: Theorem 2.5 shows that the converse to Theorem 2.2 holds for c.n.u. contractions. Theorem 2.5 says that the characteristic function, modulo coincidence, is a complete unitary invariant for c.n.u contractions. This indicates that it should be possible to recover a c.n.u contraction, up to unitary equivalence, from its characteristic function.

Recall that a C_{11} contraction is quasisimilar to a unitary operator and not similar. The following result was proved by Kerchy [3].

Theorem 2.6 If T is a c.n.u contraction and the characteristic function Θ_{T} of T is constant, then T is the orthogonal direct sum of a unilateral shift and a C_{11} contraction.

Theorem 2.6 says that a c.n.u contraction with a constant characteristic function decomposes as a direct sum of a unilateral shift and an operator quasisimilar to a unitary operator. But since T is c.n.u, the C_{11} part of T cannot be unitary, otherwise this would contradict the complete non-unitarity of T. The following result was proved by Nagy and Foias [5].

Theorem 2.7 [5]**,** (Theorem I.3.2). To every contraction T on a Hilbert space $\mathcal{H}$ there corresponds a uniquely determined decomposition of $\mathcal{H}$ into an orthogonal sum of subspaces reducing T, say $\mathcal{H} = \mathcal{M}_1 \oplus \mathcal{H}_1$, such that $\mathrm{T}_0 = \mathrm{T}|_{\mathcal{M}_1}$ is unitary and $\mathrm{T}_1 = \mathrm{T}|_{\mathcal{H}_1}$ is c.n.u. In particular, for an isometry, this canonical decomposition coincides with the von Neumann-Wold decomposition.

For a contraction T with decomposition $T = T_0 \oplus T_1$ as in Theorem 2.7, we have $D_T = 0 \oplus D_{T_1}, D_{T^*} = 0 \oplus D_{T_1^*}, \mathcal{D}_T = \mathcal{D}_{T_1}$ and $\mathcal{D}_{T^*} = \mathcal{D}_{T_1^*}$. These results lead us to the following result.

We present a simple method to show that $\Theta_T(\lambda) = \Theta_{T_1}(\lambda)$, using the notion of orthogonal projections from $\mathcal{H} = \mathcal{M}_0 \oplus \mathcal{M}_1$ onto $\mathcal{M}_0$ and $\mathcal{M}_1$, respectively.

Corollary 2.8 *Let $T \in B(\mathcal{H})$ have the decomposition $T = T_0 \oplus T_1$, with respect to the direct sum decomposition $\mathcal{H} = \mathcal{M}_0 \oplus \mathcal{M}_1$, where T_0 is unitary and T_1 is c.n.u. Then $\Theta_T(\lambda) = \Theta_{T_1}(\lambda)$.*

Proof. First, we note that $T = T_0 \oplus T_1 = T_0 P_0 + T_1 P_1$, where P_0 and P_1 are the orthogonal projections from $\mathcal{H}$ onto $\mathcal{M}_0$ and $\mathcal{M}_1$, respectively. Clearly $D_T = (I - T_1^* T_1) P_1$ and $D_{T^*} = (I - T_1^* T_1) P_1$, and consequently, $D_T = D_{T_1} P_1$ and $D_{T^*} = D_{T_1^*} P_1$. This leads to the fact that $\mathcal{D}_T = \mathcal{D}_{T_1} \subseteq \mathcal{M}_1$ and $\mathcal{D}_{T^*} = \mathcal{D}_{T_1^*} \subseteq \mathcal{M}_1$. A simple computation and an application of Theorem 2.3 using the fact that $T_0|_{\mathcal{D}_{T_1}} = 0$ and $T_0^{*n-1}|_{\mathcal{D}_{T_1}} = 0$ we get

$$\begin{array}{lcl}
\Theta_T(\lambda) & = & \Theta_{T_0 \oplus T_1}(\lambda) \\
\square & = & [-(T_0 \oplus T_1) + \lambda(D_{T_0^* \oplus T_1^*})(I - \lambda(T_0^* \oplus T_1^*))^{-1} D_{T_0 \oplus T_1}] \\
\square & = & \{-(T_0 \oplus T_1) + \displaystyle\sum_{n=1}^{\infty} \lambda^n D_{T_1^*}(T_0 \oplus T_1)^{*n-1} D_{T_1}\}|_{\mathcal{D}_{T_1}} \\
 & = & \{-T_1 + \displaystyle\sum_{n=1}^{\infty} \lambda^n D_{T_1^*} T_1^{*n-1} D_{T_1}\}|_{\mathcal{D}_{T_1}} \\
\square & = & \Theta_{T_1}(\lambda).
\end{array}$$

Theorem 2.9 *If T is a contraction, then $\Theta_T(\lambda)$ is a unitary operator on any arc of $\partial\mathbb{D}$ belonging to the resolvent set of T.*

Proof. Denote by Ω_T the set of complex numbers λ for which the operator $(I - \lambda T^*)$ is boundedly invertible. For $\lambda \in \Omega_T$ and using the Neumann series for $(I - \lambda T^*)^{-1}$, we define

$$\Theta_T(\lambda) = \{-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T\}|_{\mathcal{D}_T} \qquad (2.1)$$

$$\text{⍰} = \{-T + \textstyle\sum_{n=1}^{\infty} \lambda^n D_{T^*} T^{*n-1} D_T\}|_{\mathcal{D}_T}, \lambda \in \mathbb{D} \qquad (2.2)$$

Clearly Ω_T is an open set that contains $\mathbb{D}$ and $\Theta_T(\lambda)$ is analytic on Ω_T. Applying (2.1) to T^* as well as to T, we obtain
$\Theta_{T^*}(\lambda) = \Theta_T(\bar{\lambda}), \lambda \in \Omega_T,$

and thus in particular

$$\Theta_{T^*}(\lambda) = \Theta_{\hat{T}}(\lambda), \lambda \in \mathbb{D}$$

If $\lambda \in \partial\mathbb{D}$ (i.e. λ is a point on the unit circle) belonging to the resolvent set $\rho(T)$, then $\lambda = (\bar{\lambda})^{-1} \in \Omega_T$ and hence

$$\Theta_T(\lambda)^{-1} = \Theta_{T^*}(\lambda^{-1}) = \Theta_{T^*}(\bar{\lambda}) = \Theta_T(\lambda)^*.$$

This proves that $\Theta_T(\lambda)$ is a unitary operator.
The class of nilpotent contractions yields a natural set of examples of c.n.u contractions with polynomial characteristic functions. For example, let T be a contraction and nilpotent of order n, where $n \geq 1$. That is, $\| T \| \leq 1$ and $T^n = 0$ and $T^{n-1} \neq 0$. Therefore $D_{T^*} T^{*k} D_T = 0$ for all $k \geq n$. Using this fact, the characteristic function Θ_T of T is given by

$$\begin{aligned}\Theta_T(\lambda) &= [-T+\lambda D_{T^*}(I_{\mathcal{H}}-\lambda T^*)^{-1}D_T]|_{\mathcal{D}_T}\\ \text{⍰} &= [-T+\textstyle\sum_{k=0}^{\infty}\lambda^{k+1}D_{T^*}{T^*}^{k}D_T]|_{\mathcal{D}_T},\\ \text{⍰} &= [-T+\textstyle\sum_{k=0}^{n-1}\lambda^{k+1}D_{T^*}{T^*}^{k}D_T]|_{\mathcal{D}_T}\end{aligned}$$

for all $\lambda\in\mathbb{D}$ (see) [2]. This shows that Θ_T is a polynomial of degree less or equal to n.

Structure of c.n.u. Contractions

Recall that a pure isometry is a unilateral shift of some multiplicity and a pure co-isometry is the adjoint of a pure isometry. It has been shown in [2] that the characteristic function Θ_T of a c.n.u. contraction T on a Hilbert space $\mathcal{H}$ is a polynomial of degree $\le n$ if and only if there exist three subspaces $\mathcal{H}_1,\mathcal{M}_1,\mathcal{H}_{-1}$ of $\mathcal{H}$ such that $\mathcal{H}$ has the orthogonal decomposition $\mathcal{H}=\mathcal{H}_1\oplus\mathcal{M}_1\oplus\mathcal{H}_{-1}$ and a pure isometry $S\in B(\mathcal{H}_1)$, a nilpotent operator $N\in B(\mathcal{M}_1)$ of order n and a pure co-isometry $C\in B(\mathcal{H}_{-1})$ such that T has the following matrix representation or triangulation $T=\begin{bmatrix}S & * & *\\ 0 & N & *\\ 0 & 0 & C\end{bmatrix}$. The spaces on which one or two of the diagonal blocks acts may be $\{0\}$ and this leads to the following six degenerate cases:

$$T=\begin{bmatrix}S & *\\ 0 & N\end{bmatrix},\begin{bmatrix}S & *\\ 0 & C\end{bmatrix},\begin{bmatrix}N & *\\ 0 & C\end{bmatrix},[S],[N],[C].$$

Function models for the degenerate cases of contractions have been constructed in [2] . The case when $T=[S]$ has been solved in Theorem 2.3. The case when $T=[C]$ easily follows from Theorem 2.3 by considering S^* instead of S .

Theorem 2.10 Let $\mathrm{T}\in\mathrm{B}(\mathcal{H})$ be a c.n.u contraction with an operator-valued polynomial characteristic function Θ_{T}. Then T takes the form $\mathrm{T}=\begin{bmatrix}\mathrm{S} & *\\ 0 & \mathrm{C}\end{bmatrix}$, where S and C^* are unilateral shifts of arbitrary multiplicities if and only if $\Theta_{\mathrm{T}}(\lambda)$ coincides with $\Theta_{\mathrm{T}}(0)$, $\lambda\in\mathbb{D}$(that is the characteristic function is constant).

In Theorem 2.10, C is a co-isometry. The spaces on which the two diagonal blocks acts may be the zero space $\{0\}$. Therefore some of these entries may be absent. If T is an isometry, then the characteristic function is a constant. In this particular case, $\Theta_T(\lambda)$ and $\Theta_T(0)$ not only coincide but are equal, this also applies to a unilateral shift, since it is an isometry.

We give an examples of the case when $T=[N]$, where N has order 2,3 in $\mathbb{C}^2$ and $\mathbb{C}^3$.

Example 2.11 For $\mathrm{T}=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ acting on the Hilbert space $\mathcal{H}=\mathbb{C}^2$, it is easy to show that the 2×2-matrix-valued characteristic function of T is $\Theta_{\mathrm{T}}(\lambda)=\begin{bmatrix}0 & -1\\ \lambda^2 & 0\end{bmatrix}|_{\mathrm{span}\{\binom{1}{0}\}}$. This shows that Θ_{T} is a polynomial of degree less or equal to 2. A simple computation also shows that

$$\Theta_T(\lambda)^{-1}=\begin{bmatrix}0 & \frac{1}{\lambda^2}\\ -1 & 0\end{bmatrix}=\Theta_{T^*}(\tfrac{1}{\lambda})=\Theta_{T^*}(\bar{\lambda})=\Theta_T(\lambda)^*,$$

which agrees with Theorem 2.9. We note also that $\Theta_T(\lambda)^{-1}$ is not contractive for $\lambda\to 0$. This means that $\|\Theta_T(\lambda)^{-1}x\|\ge\|x\|$ for all $x\neq 0$ as $\lambda\to 0$.

The following example gives a c.n.u contraction which is nilpotent of order 3 but it has a constant characteristic polynomial.

Consider $T=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$. A simple computation shows that $\Theta_T(\lambda)=\Theta_T(0)=-T|_{\mathcal{D}_T}$. This shows that the characteristic function is independent of λ and is thus constant. That is, it is of degree zero.

Note that the characteristic function of these degenerate cases may be independent of the order of N. For instance, the operator $T=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ is nilpotent of order 2, but $\Theta_T(\lambda)=\begin{bmatrix}0 & -1 & 0\\ \lambda^2 & 0 & 0\\ 0 & 0 & \lambda\end{bmatrix}$.

A contractive analytic function $\Theta:\mathbb{D}\longrightarrow B(\mathcal{D}_T,\mathcal{D}_{T^*})$ is called inner if its boundary values $\Theta(e^{it})$ are isometries (or have modulus 1) almost everywhere on $\partial\mathbb{D}$.

Theorem 2.12 *Let $T\in B(\mathcal{H})$ be a c.n.u contraction. Then $T\in C_{.0}$ if and only if Θ_T is inner.*

A c.n.u. contraction T such that $\Psi(T)=0$, for some nonzero function $\Psi\in\mathbb{H}^\infty$, then $T^n\longrightarrow 0$ and ${T^*}^n\longrightarrow 0$, i.e. $T\in C_{00}$. Thus a c.n.u contraction $T\in C_{00}$ is said to be a C_0-contraction if there exists an inner function Ψ such that $\Psi(T)=0$.

We call C_0 the class of those completely non-unitary contractions T for which there exists a nonzero function $\Psi \in \mathbb{H}^\infty$ such that $\Psi(T) = 0$. Clearly $C_0 \subset C_{00}$. In Nagy and Foias [5] it is observed that for $T \in C_0$, the function Ψ in the definition can always be taken to be inner. Clearly Θ_T is inner precisely when $T^n \longrightarrow 0$ strongly(i.e. $T \in C_{0.}$).

Corollary 2.13 *Let* $T \in B(\mathcal{H})$ *be a c.n.u contraction. Then* $T \in C_{00}$ *if and only if* Θ_T *is inner.*

Proof. Suppose Θ_T is inner. Then $T^n \longrightarrow 0$ strongly, which means that $T \in C_{0.}$. Conversely, suppose $T \in C_{00}$. Then $T \in C_{0.} \cap C_{.0}$. Using Theorem 2.12 and the fact that T is a c.n.u contraction, the result follows.

If $\partial_{T^*} < \infty$, then $B(\mathcal{D}_T, \mathcal{D}_{T^*})$ is topologically isomorphic to a separable Hilbert space. Recall that the Hardy space $\mathbb{H}^2$ or $\mathbb{H}^2(\mathbb{D})$ is the space of analytic functions on $\mathbb{D}$ with square-summable Taylor coefficients; that is

$$\mathbb{H}^2(\mathbb{D}): \; = \; \{f: \mathbb{D} \longrightarrow \mathbb{C}, analytic, f(z) = \sum_{n=0}^{\infty} a_n z^n, \parallel f \parallel^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty\}$$

$$\square \; = \; \{f: \mathbb{D} \longrightarrow \mathbb{C}, analytic\text{: } \lim_{r \to 1^-} (\int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi})^{\frac{1}{2}} < \infty\}$$

Recall also the Hardy space $\mathbb{H}^\infty$ is the space of all bounded analytic functions on the unit disc $\mathbb{D}$. Clearly, $\mathbb{H}^2(\mathbb{D})$ embeds isometrically as a closed subspace of $L^2(\partial\mathbb{D})$, via

$$\sum_{n=0}^{\infty} a_n z^n \longmapsto \sum_{n=0}^{\infty} a_n e^{int},$$

where the series converges almost everywhere on $\partial\mathbb{D}$, by taking the radial limits

$$\Theta(e^{it}) = \lim_{r \to 1^-} \Theta(re^{it}), \Theta \in \mathbb{H}^2(\mathbb{D}).$$

Thus $\Theta_T: \partial\mathbb{D} \longrightarrow B(\mathcal{D}_T, \mathcal{D}_{T^*})$ belongs to the unit ball of $L^\infty(\partial\mathbb{D}, B(\mathcal{D}_T, \mathcal{D}_{T^*}))$.

We investigate some operators associated with inner functions. Let T be a contraction on a Hilbert space $\mathcal{H}$. The analytic Toeplitz operator $T_\Theta: \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ defined by $(T_\Theta f)(z) = \Theta(z)f(z)$, where $\Theta \in \mathbb{H}^\infty$ is an inner function and $z \in \mathbb{D}, , f \in \mathbb{H}^2$. This operator is similar to the shift operator S on $L^2(0, \infty)$ defined by $Sf(t) = f(t+1)$. If $\partial_T = \partial_{T^*} = 1$, these operators are unitarily equivalent.

Lemma 2.14 Let T be a contraction. The Toeplitz operator T_Θ is an isometry if and only if Θ is inner.
We denote by $\mathbb{H}^\infty(B(\mathcal{D}_T, \mathcal{D}_{T^*}))$ the operator Hardy class of bounded analytic functions whose values are bounded operators from $\mathcal{D}_T$ to $\mathcal{D}_{T^*}$.

Lemma 2.15 Let T be a c.n.u. contraction. The Toeplitz operator T_Θ is a unilateral shift if and only if Θ is a non-constant inner function.

Theorem 2.16 If $\Theta_\mathrm{T} \in \mathbb{H}^\infty(\mathrm{B}(\mathcal{D}_\mathrm{T}, \mathcal{D}_{\mathrm{T}^*}))$ is an inner function and is onto then Θ_T is a unitary constant.

Proof. Since $\Theta_T: \mathcal{D}_T \longrightarrow \mathcal{D}_{T^*}$ is onto, we have that $\Theta_T \mathcal{D}_T = \mathcal{D}_{T^*}$. Thus, the analytic Toeplitz operator T_Θ associated with T is a unitary operator. Thus, for every $x \in \mathcal{D}_T$ the function $(T_\Theta x)(z) = \Theta(z)x$ is constant and hence $\Theta_T(\lambda) = \Theta_T(0)$. Since Θ_T is inner, we conclude that $\Theta_T(0)$ must be an isometry. Since $\Theta_T(0): \mathcal{D}_T \longrightarrow \mathcal{D}_{T^*}$ is onto, $\Theta_T(0)\mathcal{D}_T = \mathcal{D}_{T^*}$, and so $\Theta_T(0)$ is an onto isometry, which means that it is a unitary constant.

Proposition 2.17 Let $\mathrm{T} \in \mathrm{B}(\mathcal{H})$ be a contraction of class $\mathrm{C}_{.0}$ such that $\mathrm{d}_{\mathrm{T}^*} = 1$. Then either one of the following statements hold:

(i). T is unitarily equivalent to the shift S.
(ii). T is unitarily equivalent to the shift S_Θ, for some nonconstant inner function Θ.
If T is a c.n.u. contraction and $\Theta, \Psi \in L^\infty$, then $T_\Theta T_\Psi = S_{\Psi\Theta}$, where $T_\Theta: \mathbb{H}^2 \longrightarrow \mathbb{H}^2$ defined by $T_\Theta f = P(\Theta f)$, where $P: L^2(\partial\mathbb{D}) \longrightarrow \mathbb{H}^2$ is the Riesz projection. As seen earlier, if $\Theta \in \mathbb{H}^\infty$, the Toeplitz operator T_Θ is just the multiplication operator.

Theorem 2.18 *Let* T *be a c.n.u. contraction. Then* T_Θ *is idempotent if and only if* $\Theta_T = 0$ *or* $\Theta_T = I$.

Example. $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is in C_{00} and hence $\Theta_T(\lambda) = \begin{bmatrix} 0 & -1 \\ \lambda^2 & 0 \end{bmatrix}$ computed earlier is inner. Clearly, $\Theta(e^{it})^* \Theta(e^{it}) = \begin{bmatrix} 0 & e^{-2it} \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ e^{2it} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Thus, $\Theta(e^{it})$ is isometric.

Theorem 2.19 *Let* $T \in B(\mathcal{H})$ *be a c.n.u contraction with* $d_T = d_{T^*} = 1$. *Then* Θ_T *is scalar-valued.*

In Theorem 2.19, we may identify the spaces $\mathcal{D}_T$ and $\mathcal{D}_{T^*}$ with $\mathbb{C}$.

Similarity, Quasisimilarity and Metric Equivalence of c.n.u. Contractions

In this section we give necessary and sufficient conditions for two c.n.u contractions to be similar, metrically equivalent or Quasisimilarity. We try to characterize similarity, metric equivalence and quasi-similarity of contractions in terms of their characteristic functions. We note that unlike unitarily equivalent c.n.u contractions, the characteristic functions of similar c.n.u contractions need not be similar or coincide . To see this, the operators $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $S = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ are similar c.n.u contractions (with similarity transformation $X = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$), but their respective characteristic functions, $\Theta_T(\lambda) = \begin{bmatrix} 0 & -1 \\ \lambda^2 & 0 \end{bmatrix}$ and $\Theta_S(\lambda) = \begin{bmatrix} \frac{\sqrt{3}}{2}\lambda & -\frac{1}{2} \\ -\frac{1}{2}\lambda^2 & \frac{\sqrt{3}}{2}\lambda \end{bmatrix}$ are not similar/do not coincide for all values of $\lambda \in \mathbb{D}$. Note also that there cannot exist non-similar contractions with identically the same characteristic functions.

Theorem 2.20 *If* $T \in B(\mathcal{H})$ *is unitarily equivalent to a unitary operator, then it is unitary.*

Proof. Suppose that $T = U^*SU$, where U and S are unitary. The result follows from

$$T^*T = U^*S^*SU = I = U^*SS^*U = TT^*.$$

Note that similarity to a unitary operator need not imply unitary.
Is complete non-unitarity invariant under similarity, Quasisimilarity or metric equivalence?

Theorem 2.21 If T ∈ B($\mathcal{H}$) is metrically equivalent to a unitary operator, then it is isometric.
Note that complete non-unitarity is not invariant under metric equivalence. For instance, the unilateral shift and the identity operator in $\mathcal{H} = \ell^2$ are metrically equivalent, but the identity operator is not c.n.u.

Theorem 2.22 If S and T are metrically equivalent c.n.u contractions, then $\Theta_{|S|}(\lambda) = \Theta_{|T|}(\lambda)$.

Proof. The result follows easily from the fact that metric equivalence of T and S implies $|S| = |T|$.
Corollary 2.23 Suppose S and T are self-adjoint metrically equivalent c.n.u contractions. Then then $\Theta_S(\lambda) = \Theta_T(\lambda)$.

Theorem 2.24 If T is similar to a c.n.u contraction and if its spectrum has zero Lebesgue measure, then $\sup_{|\lambda|<1} \| \Theta_T(\lambda)^{-1} \| = \infty$.

Theorem 2.25 *T is an invertible contraction if and only if $\Theta_T(0)$ is invertible.*

Characteristic Function as an Operator Pencil

A polynomial operator pencil, also known as an operator polynomial, is an expression of the form

$$L(\lambda) = \lambda^n A_n + \lambda^{n-1} A_{n-1} + \cdots + A_0,$$

where the A_k are linear operators in a Hilbert space and $\lambda \in \mathbb{C}$ is the spectral parameter.

The characteristic function of a c.n.u. contraction can be expressed as

$$\Theta_T(\lambda) = \{-T + \lambda D_{T^*}(I_{\mathcal{H}} - \lambda T^*)^{-1} D_T\}|_{\mathcal{D}_T} = \{-T + \lambda U D_T\}|_{\mathcal{D}_T}, \quad (3.1)$$

where $U = D_{T^*}(I_{\mathcal{H}} - \lambda T^*)^{-1}$ and for all $\lambda \in \mathbb{D}$.

Remark. In equation (3.1) the characteristic function is presented as a linear operator pencil.

A contraction $T \in B(\mathcal{H})$ is called a pure contraction if $\| Tx \| < \| x \|$, for all $x \in \mathcal{H}$. Clearly, a pure contraction is completely non-unitary.

Theorem 3.1 *Let $T \in B(\mathcal{H})$ be a c.n.u. pure contraction. Then the operator*

$$U = D_{T^*}(I_{\mathcal{H}} - \lambda T^*)^{-1}$$

and for all $\lambda \in \mathbb{D}$ is unitary.

Proof. The operator $U: \mathcal{D}_T \longrightarrow \mathcal{D}_{T^*}$ is defined by $Ux(\lambda) = D_{T^*}(I_{\mathcal{H}} - \lambda T^*)^{-1} x(\lambda)$, for all $x \in \mathcal{D}_T$ and $\lambda \in \mathbb{D}$. It suffices to prove that T is injective and surjective.
Using the Neumann series expansion, we have that

$$Ux(\lambda) = \sum_{k=0}^{\infty} \lambda^k D_{T^*} T^{*k} = \lambda^0 D_{T^*} x + \lambda^1 D_{T^*} T^* x + \lambda^2 D_{T^*} T^{*2} x + \cdots$$

Therefore, taking the $\mathcal{D}_{T^*}$-norm both sides and using the Pythagoras Theorem and the fact that T is pure, we have

$$\begin{array}{lll} \| Ux \|^2 & = & \| D_{T^*}x \|^2 + \| \lambda^1 D_{T^*}T^*x \| + \| \lambda^2 D_{T^*}T^{*2}x \| + \cdots \\ & = & \| x \|^2 - \| \lambda^1 T^*x \|^2 + \| \lambda^1 T^*x \|^2 - \| \lambda^2 (T^*)^2 x \|^2 + \| \lambda^2 (T^*)^2 x \|^2 - \cdots \\ \text{⍰} & = & \| x \|^2 - \lim\limits_{n\to\infty} \| \lambda^n T^{*n} x \|^2 \\ \text{⍰} & = & \| x \|^2 \end{array}$$

This proves that U is an isometry, and hence injective.
It is not easy to show that $Ran(U) = \mathcal{D}_{T^*}$. However, it is easy to show that T is invertible. This follows from the fact that since T is a pure contraction, so is T^* (since $\| T^* \| = \| T \| < 1$) and so D_{T^*} is invertible. By hypothesis $(I_{\mathcal{H}} - \lambda T^*)^{-1}$ is also invertible, which means that U is an invertible isometry. This establishes the claim.

Theorem 3.2 Let T be a nilpotent c.n.u contraction. Then $\Theta_{|T|}(\lambda) = \mathrm{diag}(0, \sum_{k=0}^{\infty} \lambda^k) = \mathrm{diag}(0, \frac{1}{1-\lambda})$.

Example. For the operator $T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, a simple calculation gives that $|T| = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, and $\Theta_{|T|}(\lambda) = \begin{bmatrix} 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \lambda^k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{1-\lambda} \end{bmatrix}$.

We note that there is a relationship between T and $\Theta_T(\lambda)$. In fact,

$\sup\{\| \Theta_T(\lambda) \| : \| \lambda \| < 1\} = C = \| T \|$.

Theorem 3.3 A contraction operator $\mathrm{T} \in \mathrm{B}(\mathcal{H})$ is similar to a unitary operator if and only if $\Theta_{\mathrm{T}}(\lambda)$ is an invertible operator for every $\lambda \in \mathbb{D}$.

Theorem 3.4 *If* $\frac{1}{\lambda} \notin \sigma(T^*)$, *then* $\Theta_T(\lambda)^* = \Theta_{T^*}(\bar{\lambda})$. *If in addition* $\lambda \notin \sigma(T)$, *then* $\Theta_T(\lambda)$ *is invertible and* $\Theta_T(\lambda)^{-1} = \Theta_{T^*}(\frac{1}{\lambda})$.

Proof. The first claim is straightforward from definition. For the second claim, we first note that

$$\Theta_{T^*}(\tfrac{1}{\lambda}) D_{T^*} = D_T (I - \tfrac{1}{\lambda} T)^{-1} (\tfrac{1}{\lambda} I - T^*).$$

Multiplying from the left by $\Theta_T(\lambda)$ and simplifying, we have

$$\Theta_T(\lambda) \Theta_{T^*}(\tfrac{1}{\lambda}) D_{T^*} = D_{T^*}.$$

By symmetry, we also have

$$\Theta_{T^*}(\tfrac{1}{\lambda}) \Theta_T(\lambda) D_T = D_T.$$

Theorem 3.5 A contraction $\mathrm{T} \in \mathrm{B}(\mathcal{H})$ is Fredholm if and only if $\Theta_{\mathrm{T}}(0)$ is, and $\mathrm{ind}(\mathrm{T}) = \partial_{\mathrm{T}} - \partial_{\mathrm{T}^*}$.

Theorem 3.6 For any contraction $\mathrm{T} \in \mathrm{B}(\mathcal{H})$ the operators D_{T} and $\mathrm{D}_{\Theta_{\mathrm{T}}(0)}$ are unitarily equivalent.

Proof. Obvious from the definition of Θ_T.

A unilateral shift is the classical example of non-unitary isometry. In fact, they are prototypes of non-unitary isometries. Any truncated shift operator is nilpotent.

Theorem 3.7 [4] (Problem 5.3) Two shifts are unitarily equivalent if and only if they have the same multiplicity.

The following result is a consequence of Theorem 3.7.

Corollary 3.8 Two c.n.u contractions have same multiplicity if and only if their characteristic functions coincide.

Discussion

Contractive operator-valued analytic functions play an important role in operator theory and serve as a bridge between operator theory and function theory in terms of system realization theory, interpolation theory and scattering and transfer function theory in classical and quantum physics and linear time invariant systems (cf.) [1, 5, 6]. Clearly a contraction operator can be investigated in details in terms of its Sz.-Nagy-Foia*s* characteristic function.

Conclusion

This study explores key properties and behaviors of contractions in Hilbert spaces, focusing on their characteristic functions and decompositions. Through the application of Sz.-Nagy-Foias theory, the work demonstrates that completely non-unitary (c.n.u.)

contractions exhibit unique characteristic functions that provide a comprehensive understanding of their structure. Theorems presented highlight the relationship between characteristic functions and operator equivalence, offering insights into isometric, unitary, and nilpotent operators. Ultimately, the study contributes to a deeper understanding of operator theory by linking analytical functions with operator behaviors and classifications.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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